Journal Club for August 2023: Attractors in stressed granular materials

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1. Introduction

An attractor is a set of states toward which a dynamic system tends to evolve, for a wide variety of starting conditions of the system (Wikipedia). System values that get close enough to the attractor values remain close even if slightly disturbed. A famous example is the Lorenz attractor: A system of ordinary differential equations originally proposed to describe the atmospheric convection (Lorenz, 1963) exhibits chaotic behaviors, i.e., slightly different initial conditions lead to fundamentally different solutions shortly after the start of the evolution. However, they all appeared to be attracted by but never reach certain points in the phase space, generating the well-known butterfly shape (Fig. 1). The notion of an attractor offers an important means to characterize chaotic, nonlinear, and complex systems.

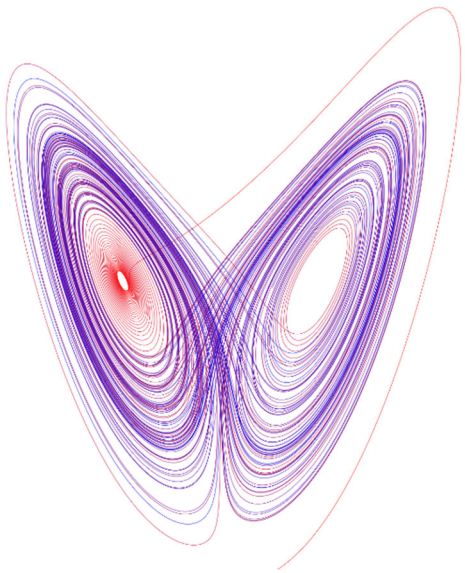

Figure 1: A solution in the Lorenz attractor (Wikipedia)

Granular materials are complex. They consist of many simple subunits (i.e., grains) interacting with each other in their close neighborhood through friction, rotation, interlocking, crushing, and abrasion. Emerged from these lower-scale interactions are the nonlinear and adaptative mechanical response of granular materials: Its stress-strain relation is highly nonlinear and history dependent; The fabric of granular assemblies (e.g., the orientations of contact, grain, and void) constantly evolves in adaptation to the applied external stresses; In high-stress regime where grain breakage can occur, the grain size and shape can also self-organize to gain energetic and survival advantages. Due to the rich nonlinear interactions between grains, a predictive constitutive theory based on the sole properties of the individual particles has not been proposed yet, although computer programs resolving the grain-scale interactions (e.g., discrete element

method) can successfully mimic the macroscopic mechanical behaviors of granular materials, i.e., the case of weak emergence (Bedau, 1997).

Just like many other complex systems, granular materials possess several distinct attractors that can be regarded as their "genes". These attractors are extremely useful in constitutive theories to portrait the macroscopic mechanical response of granular materials. In this article, I will attempt a brief review of some experimentally and numerically identified attractors in granular systems. The goal is to summarize the various attractor-like concepts studied in soil mechanics, granular physics, and engineering mechanics communities, and hopefully to spark some discussions and inspirations among the readers.

2. Critical State

The first attractor-like concept found for sheared granular materials is the Critical State Line (CSL) defined in the void ratio e – mean stress p – deviatoric stress q space (Fig. 2). It states that all granular materials, when monotonically sheared to large strains, will reach a stationary void ratio and stress ratio in the *e-p-q* space. The critical state condition can be mathematically stated by:

$$q/p = M;\ e = e_c(p) \tag{1}$$

where e is the void ratio; p the mean stress (effective mean stress for saturated soils); q the deviatoric stress.

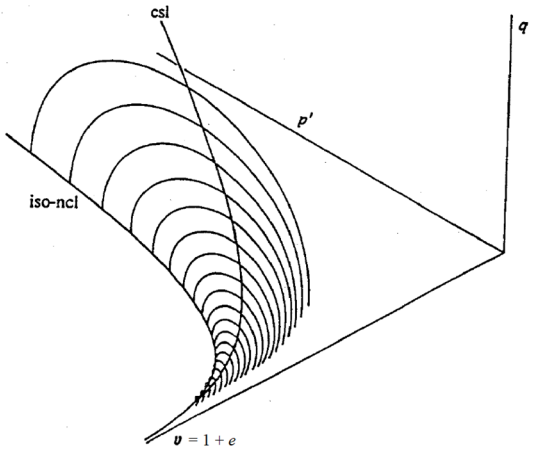

Figure 2: Critical state line in the *e-p-q* space (Wood, 1990).

Casagrande (1936) was the first to note that 'every cohesionless soil has a certain critical, in which state it can undergo any amount of deformation or actual flow without volume change.' Wroth (1958) conducted simple shear test on 1-mm diameter steel balls and confirmed the emergence of a CSL in the *e-p-q* space. The existence of CSL was then systematically identified from experiments on reconstituted clays (Alan Bishop and Henkel, 1957; Roscoe et al., 1958). Subsequently, more supporting data were collected on natural sand (Verdugo and Ishihara, 1996; Vesic and Clough, 1968) and rockfill (Marachi et al., 1972). The uniqueness of CSL was also confirmed by discrete element method (DEM) simulations (Huang et al., 2014; Nguyen et al., 2018).

Constitutive modeling. The universality and robustness of CSL drove the establishment of the Critical State Soil Mechanics (CSSM) theory (Schofield and Wroth, 1968; Wood, 1990). CSSM has inspired numerous soil constitutive models that utilize CSL as their fundamental building block. Most of them are constructed following the architecture of elastoplasticity (Gajo and Wood, 1999; Jefferies, 1993; Manzari and Dafalias, 1997). Some others are formulated under hypoplasticity (Niemunis and Herle, 1997; von Wolffersdorff,

1996), thermodynamics with internal variables (TIV) (Collins and Muhunthan, 2003; Houlsby and Puzrin, 2007), or hydrodynamic theory (Jiang and Liu, 2015).

3. Critical Fabric

There is a growing awareness that a more complete definition of critical state needs to make reference to the internal structure of granular materials. In fact, it is possible to show that the difference between the actual void ratio and the critical state void ratio does not fully determine the stress-strain evolution of the system (Theocharis et al., 2019). Many studies found that sands prepared to the same initial void ratio respond differently under different shearing modes (Mooney et al., 1998; Vaid and Thomas, 1995; Yoshimine et al., 1998), implying the significant impact of fabric anisotropy on the behavior of granular soils. Li and Dafalias (2012) proposed that, in addition to the void ratio and stress ratio, the second-rank fabric tensor must also reach its critical value at critical state, i.e.:

$$q/p = M;\ e = e_c(p);\ F = F_c \tag{2}$$

where F is the normalized deviatoric fabric tensor (i.e., $\mathrm{tr}(F) = 0$). This framework does not constrain the definition of fabric tensors nor limit the form of constitutive laws. Therefore, it is regarded as an extension of the original CSSM and referred to as the Anisotropic Critical State Theory (ACST).

Many DEM and laboratory experiments were followed to test the hypothesis of a unique fabric attractor for granular soils. In most of these works, granular microstructure is represented by a traceless second-order fabric tensor defined through either contact norms, void vectors, or the particle orientations. Fu and Dafalias (2011) and Wang et al. (2017) conducted two-dimensional (2D) DEM simulations of elongated particle assemblies, confirming that a unique steady state is reached for each of these three fabric tensor definitions, irrespective of the initial void ratio and the orientation and intensity of fabric anisotropy. Zhao and Guo (2013) conducted a series of true triaxial three-dimensional (3D) DEM tests and reported that the contact-based fabric tensor reaches a well-defined ultimate envelop in the principal fabric space, the shape of which is similar but reciprocal to the critical stress envelop along the deviatoric plane. They further demonstrated that the first joint invariant of the stress and fabric tensors at critical state K_c is independent of Lode angle. Nguyen et al. (2016, 2018) conducted three-dimensional (3D) DEM triaxial tests and showed that the deviatoric component of contact-based fabric tensor at critical state is uniquely related to p and e. In addition, they found that the coordination number CN at critical state is only a function of p and independent of drainage conditions and consolidation methods. Kruyt and Rothenburg (2014) through 2D DEM studies demonstrated that CN and contact-based fabric anisotropy A at critical state are functions of the interparticle friction and confining pressure. When plotting these fabric states in the "fabric space" (i.e., A vs. CN plot), a critical fabric line can be identified for the studied granular system. Zhu et al. (2016) and Deng et al. (2021) confirmed that the critical state void ratio and fabric descriptors achieved in a shear band at localized failure are identical to those obtained through a diffusive failure pattern.

Recently, my group repeated the true triaxial DEM simulations of Zhao and Guo (2013) but extended the simulation to extremely loose sand under undrained shear and employed a new fabric tensor definition that preserves the hydrostatic component (instead of the traceless tensor used in Eq. (2) and other works)(Wen and Zhang, 2022a). An interesting finding is that even liquefied (or "unjammed" in granular physics community), apparently structureless granular assemblies exhibit a unique critical fabric after sufficient shear. These critical fabric data in conjunction with that of jammed granular soils makes a complete 3D critical fabric surface (Fig. 3), offering a well-defined, mathematically continuous attractor for granular

materials in the principal fabric space. Building upon this initial discovery, we examined the critical fabric data of a variety of granular packings that are initially dense vs. loose, isotropic vs. anisotropic, polydisperse vs. bi-disperse, with contact law being Hertz Mindlin vs. linear elastic, shear mode being simple shear vs. triaxial, monotonic vs. cyclic, boundary-driven shear vs. athermal quasistatic shear with Lees-Edwards boundary (Wen and Zhang, 2022a, b, 2023). All systems exhibit similar CFS at critical state, confirming the robustness of this attractor and its insensitivity to protocol/ system variations.

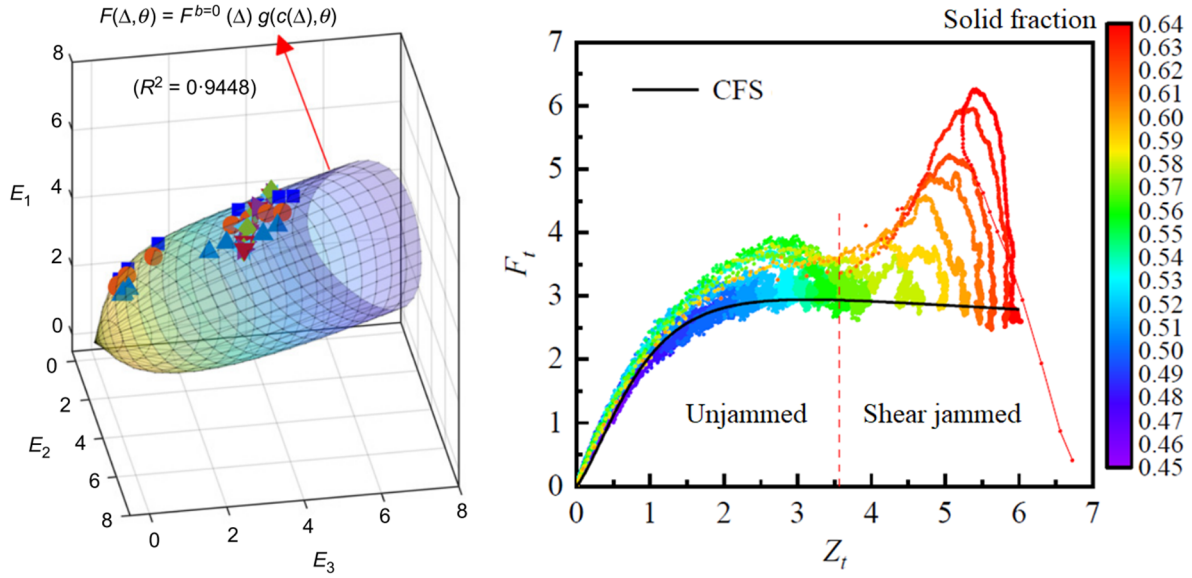

Figure 3: Left: critical fabric surface identified through true triaxial shear on a polydisperse assembly (Wen and Zhang, 2022a). Data points in the figure represent the critical fabric data of samples with different initial states. Right: fabric evolution paths of initially unjammed, bi-disperse granular samples (Wen and Zhang, 2023). Depending on the initial density (indicated by the color bar), some samples remain unjammed throughout the shearing process, while others develop certain shear stress and are thus considered to be shear jammed. All fabric paths converge to the CFS at steady state.

In contrast to DEM, in-situ experimental determination of fabric evolution requires much more effort in terms of test designs and data postprocessing. Oda (1972a, b) studied the cross sections of resin-impregnated sand specimens and found a strong correlation between fabric anisotropy and stress ratio of triaxially loaded sand specimens. A more common method of quantifying material microstructure nowadays is the in-situ X-ray microtomography technique (X-μCT) (Andò et al., 2012; Desrues et al., 1996). In this approach, miniature triaxial or oedometric tests were conducted on an X-μCT platform. X-ray scanning and mechanical loading were alternated to obtain snapshots of the specimen's internal structure throughout the deformation process. This X-ray image dataset then requires tremendous effort in post processing to convert to physically sensible 3D models of the granular assembly, through which specimen's fabric statistics can be extracted. Using X-μCT, Imseeh et al. (2018) observed that the contact-based fabric tensor indeed reaches a steady-state value coinciding with the critical state condition in triaxial tests. However, the evolution of fabric towards such an attractor is found to be non-monotonic for some specimens. Wiebicke et al. (2020) and Zhao et al. (2021) distinguished fabric evolution within and outside of the shear band, and found that fabric anisotropy and coordination number approach unique values at large strain (within the band), irrespective of the initial fabric void ratio. In summary, both DEM and experimental findings support the existence of an attractor in the fabric space for granular materials.

Constitutive modeling. Inspired by these findings, the community is now seeing a new round of developments on sand constitutive models acknowledging fabric evolution (Dafalias and Taiebat, 2016;

Papadimitriou et al., 2019; Petalas et al., 2020; Tasiopoulou and Gerolymos, 2016; Wang et al., 2021; Zhang et al., 2020; Zhao and Gao, 2015). It is worth noting that most experimental and DEM studies have focused on fabric descriptions based on contact normal vectors. It is reasonable to believe that other fabric descriptors such as those defined on particle and void cell orientations also converge to steady state values at large shear strains. However, there is a scarcity of experimental data to substantiate this speculation or quantifying their evolution pattern towards the critical state.

4. Grain Size

The grain size distribution (GSD) of natural soil is constantly evolving due to crushing, agglomeration, mixing, and segregation (Foley, 2018; Johnson et al., 2012; Wang et al., 2002). Understanding the grain size dynamics during these processes is of great engineering and scientific significances. In geotechnical engineering, grain size distribution provides the main indices characterizing and classifying sands. It has first-order impacts on the shear strength, deformability, hydraulic conductivity, and erodibility of granular soils (Kenney and Lau, 1985; Singh et al., 2021; Wood and Maeda, 2008; Yang and Gu, 2013). In geoscience, grain size distribution encodes the history of earth's or other planets' surface (Anderson and Bunas, 1993; McKay et al., 1974). Each process of crushing, agglomeration, mixing, and segregation drive a different mode of GSD evolution, and thus difficult to identify universal trends if they are discussed altogether. Herein, this article will focus on the grain size evolution dominated by grain crushing, usually under high stress situations that can be seen at many length scales (Fig. 4).

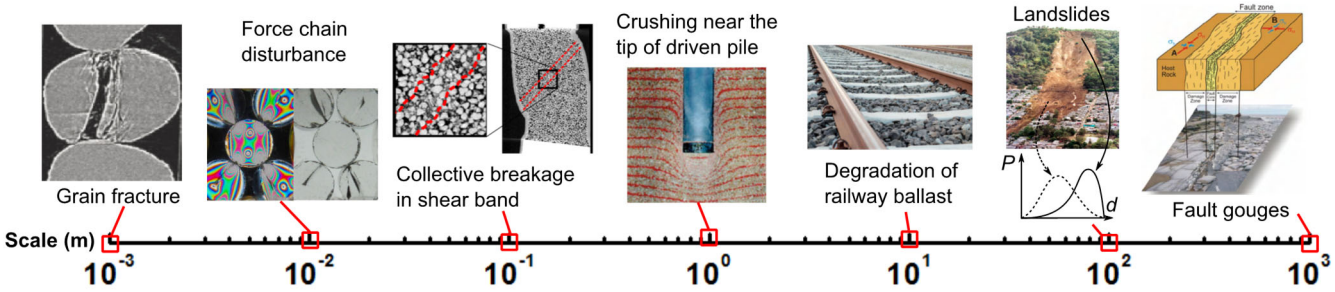

Figure 4: Grain breakage across scales.

Quantitative descriptions of the degree of grain crushing and tracking grain size evolution was mainly attempted in geotechnical engineering and geomorphology communiteis. Hardin (1985) hypothesized that all particles in a sample of soil could be crushed to fines (particles with size less than 0.074 mm) under sufficiently high stresses. This was, however, not support by the observation that fault gouge materials which have experienced extreme compression and shearing exhibit a self-similar GSD characterized by a fractal dimension of 2.6 (Sammis et al., 1987), corroborating the fragmentation theory of Turcotte (1986):

$$N(x > d) = Ad^{-\alpha} \tag{3}$$

where N is the number of particles with diameter larger than d; A is a constant of proportionality; α is the fractal dimension. The existence of a GSD attractor described by Eq. (3) with a universal fractal dimension (2.5~2.7 for 3D, 1~1.3 for 2D) for severely stressed granular materials was later supported by a boom of evidence from laboratory investigations (Coop et al., 2004; McDowell and Bolton, 1998; Nakata et al., 2001a), DEM simulations (Ben-Nun and Einav, 2010; McDowell and de Bono, 2013), and geological observations (Billi, 2005; Marone and Scholz, 1989). See Fig. 5 (left) for an example. It is worth highlighting that the ultimate fractal GSD appeared to be insensitive to minor alterantions of the initial properties of the packing nor the different modes of fragmentation (Ben-Nun and Einav, 2010), hinting that

the collective breakage of grain assembly is a robust self-organized process (Bak, 2013). In fact, Fig.5 (left) resembles the rank vs. frequency characteristics of many other complex systems that exibits self-organization, such as city size (Zipf's law), earthquake (Gutenberg-Richter law), etc. See Fig. 5 (right) for example.

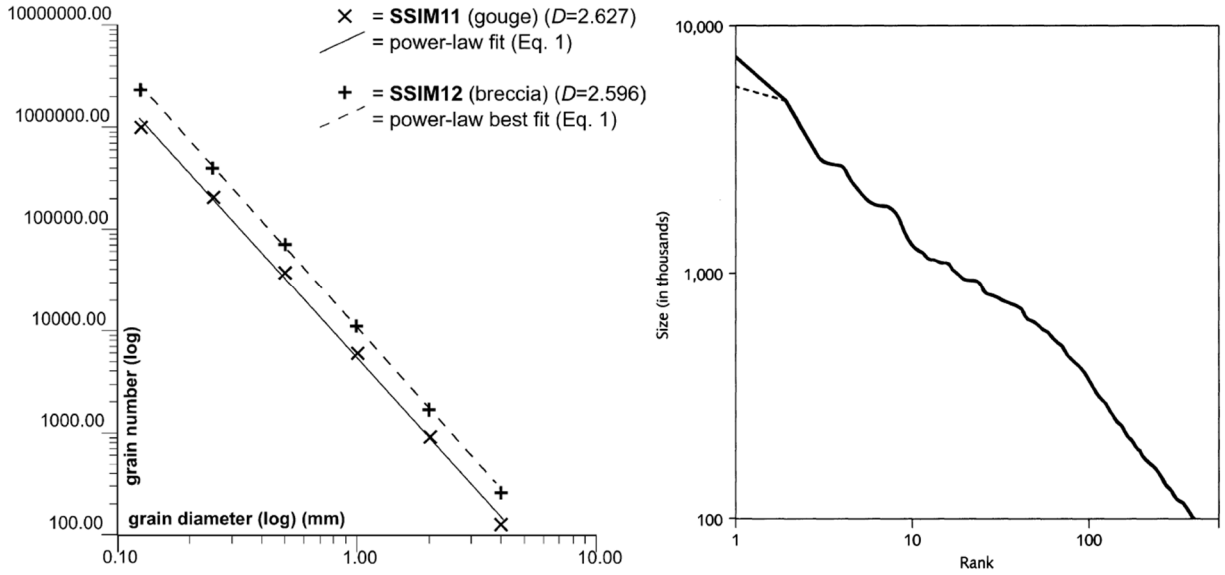

Figure 5: Left: GSD of gouge and breccia materials collected from the Mattinata fault (Billi, 2005). Right: Number of cities in which the population exceeds a given size or, equivalently, the relative ranking of cities vs. their population around year 1920 (Bak, 2013; Zipf, 2016).

The pattern of GSD evolution can be also viewed mechanistically as a result of dynamic competition between two micromechanisms: (1) smaller particles can withstand higher deviatoric stress and thus are more difficult to break (Kendall, 1978; McDowell, 2001; Nakata et al., 2001b); (2) during collective breakage, large particles get surrounded and supported by smaller particles (i.e., the so-called cushioning effect) and become less likely to break (Ben-Nun et al., 2010; Tsoungui et al., 1999). This explains the existence of an ultimate GSD and packing configuration where the likelihood of crushing of any of the particles in the system are equal and asymptotically approaches zero as confining stress increases.

It is worth to mention several recent observations that may challenge the universality of the ultimate GSD. Specifically, gap-graded soils seem to "remember" their initial GSD even after being loaded to very large stress levels (Zhang and Baudet, 2013; Zhang et al., 2017) (Fig. 6 left). Whether such a result is a natural consequence of self-organized crushing, or there are some unknown mechanisms that interrupt such self-organization for gap-graded soils, is currently not clear. It has also been pointed out that, although the ultimate GSDs generated from excessive shearing and high-pressure compression are both fractal, they appear to have different fractal dimensions even for the same sand (Miao and Airey, 2013). Gradings resulting from shear appear to be overall finer than those from compression (Fig. 6 right), which may be explained by the higher mobility of grains during shearing. More data and investigations are needed to further understand these deviations.

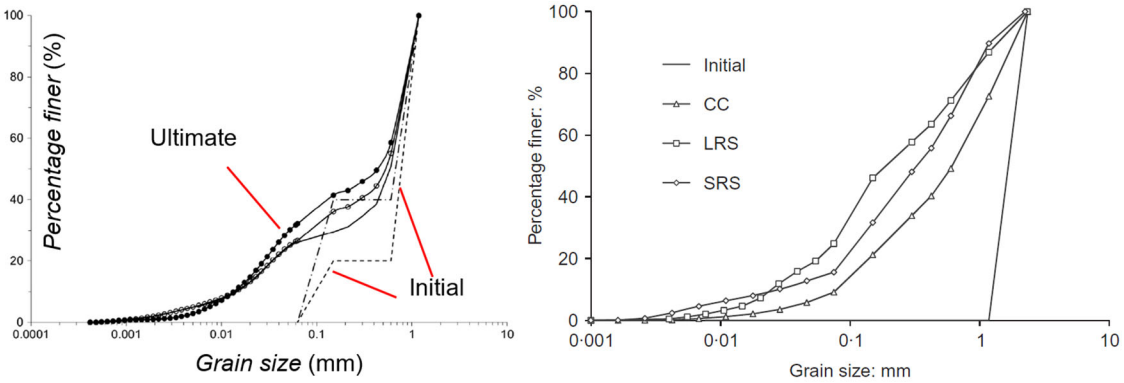

Figure 6: Left: Gap-graded soils do not exhibit a mono-fractal GSD after crushing (Zhang et al., 2017). Right: samples stressed under oedometric compression and ring shear tests develop different ultimate GSDs (Miao and Airey, 2013).

Constitutive modeling. The identification of an attractor in the GSD plane has facilitated the development of many continuum models for crushable granular materials. The leading theory in this regard is the Continuum Breakage Mechanics (CBM) (Einav, 2007a, b). The identified GSD attractor is used to define a new internal state variable called *breakage* that varies from 0 to 1 in the process of crushing (Fig. 7). Such enrichment allows the coupling of the energy, microstructure, and stress-strain response of the granular assembly. The thermomechanical formulation of CBM bears some similarity with continuum damage mechanics (CDM) for brittle solids and is also rooted in the theory of linear elastic fracture mechanics (Einav, 2007c). Within the framework of CBM, a variety of mechanical and physical properties of crushable granular materials are now possibly modelled in quantitative manner. To name a few, these models cover topics on environment-dependent fragmentation of rock aggregates (Shen and Buscarnera, 2022a; Zhang and Buscarnera, 2018), breakage-induced creep and relaxation (Alaei et al., 2021; Zhang and Buscarnera, 2017), high-strain rate comminution and penetration (Kuwik et al., 2022), grain-size dependent yielding (Zhang et al., 2016), water retention and permeability evolution (Esna Ashari et al., 2018; Gao et al., 2016; Singh et al., 2021), strain localization in granular rocks (Collins-Craft et al., 2020; Das et al., 2013; Nguyen and Einav, 2010; Tengattini et al., 2014), anisotropy and fabric-dependent breakage (Marinelli and Buscarnera, 2019; Shen and Buscarnera, 2022b). CBM models that acknowledges or predicts other attractors such as the critical state line in *e-p-q* space were also proposed (Tengattini et al., 2016).

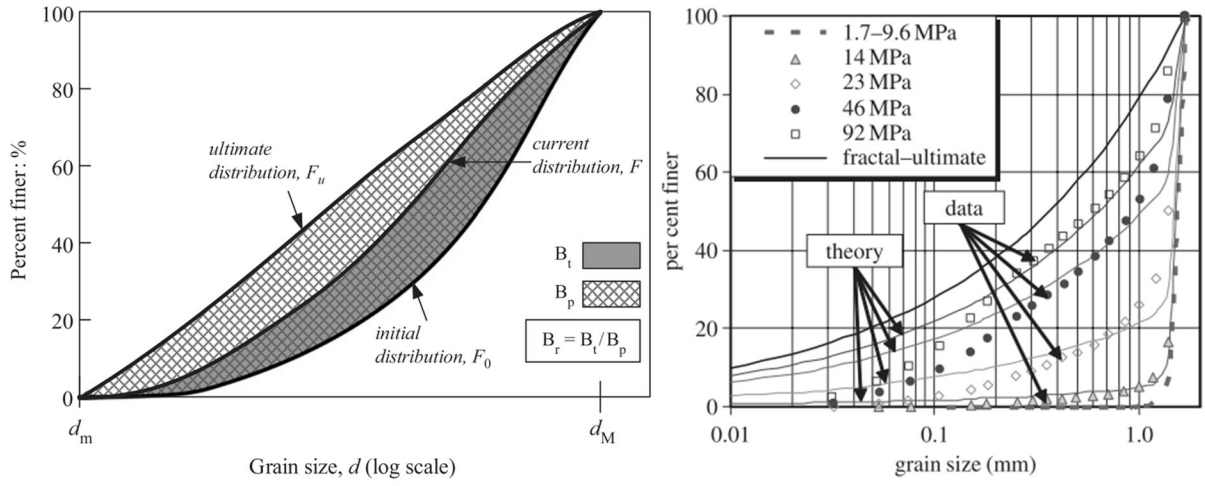

Figure 7: Left: Definition of breakage variable *B*. Right: predicted GSD vs. experimental data. (Einav, 2007a)

5. Grain Shape

Grain shape coevolves with grain size during fragmentation. Increasing evidence has pointed towards the existence of an attractor for the grain shape, in complementary to the GSD attractor, for crushed granular materials. Admittedly, research on grain shape evolution in fragmentation/ comminution is still at early stage compared to GSD evolution studies, and many conclusions may not be definite and require further validations. The situation is also complicated by the large number of grain shape descriptors proposed in the past (Anusree and Latha, 2023). This section will attempt to summarize a few studies in this regard and briefly introduce their implications to constitutive modelling of granular materials.

In the broad context of fragmentation dynamics, Domokos et al. (2015) found that rock fragments, whether generated through slowly evolving weathering or from rapid breakup induced by explosion and hammering, exhibits a self-similar shape distribution quantified by surface-to-volume ratio (Fig. 8). The dominating elongation ratio and the flatness ratio of fragmented particles exhibit dependency on grain size, i.e., larger particles tend to be more rounded. The existence of such shape attractor is elegantly explained through a discrete stochastic model of fragmentation (Domokos et al., 2020). Other grain shape attractors created by different mechanical processes and geological settings from river pebbles (Novák-Szabó et al., 2018) to boulders on asteroids (Michikami et al., 2010) have also been identified in the literature.

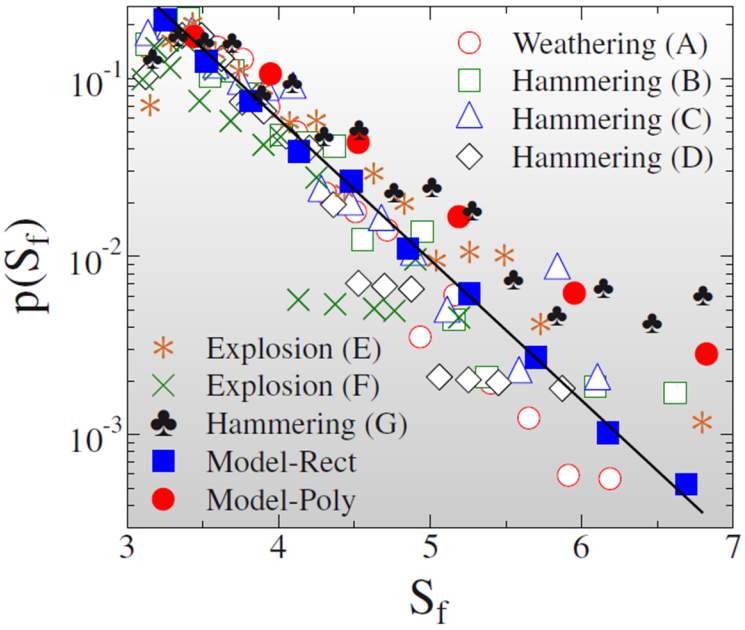

Figure 8: Probability distribution of the shape parameter for fragments generated through weathering and hammering (Domokos et al., 2015)

The universality of grain shape can be also observed from granular assemblies collectively crushed under high-pressure compression and shear. Seo et al. (2020) conducted oedometric compression in conjunction with in-situ X-μCT on two quartz sands with different grain morphologies. They found that continuous compression can mitigate morphological differences, especially when the stress is sufficient to induce pervasive breakage (See Fig. 9). Miao and Airey (2013) studied the ultimate states of a carbonate sand under different stress conditions. Both GSD and grain shape evolves towards a steady state after sufficient loading. However, samples subjected to continuous shearing (through a ring shear device) produces an ultimate grain shape that has a slightly higher aspect ratio than that obtained from high-pressure oedometric compression tests. Ueda et al. (2013) performed 2D DEM simulation of oedometric compression tests on granular assemblies with initial shapes ranging from perfect circle to elongated hexagon. They observed

that all samples arrived at a stable aspect ratio after crushed to near ultimate state, despite the presence of drastically different crushing modes such as cleavage destruction, bending fracture, and edge abrasion (See Fig. 9). Through a novel hybrid peridynamics (PD) and non-smooth contact dynamics (NSCD) simulation of oedometric crushing process, Zhu and Zhao (2021) observed that the distributions of several shape factors (elongation, flatness, aspect ratio) approach to a steady profile which can be approximated by a normal or Weibull distribution, accompanied with the reduction of median grain sizes. Ma et al. (2019) conducted a combined finite and discrete element method (FDEM) study of the same process and reported similar conclusions for other shape descriptors such as surface-volume ratio, sphericity, and convexity. Their simulation indicates that the dominate form of grain breakage is the splitting of particles into several fragments of similar size at the onset of yielding, while it changes to the abrasion of local asperities with further increase of stress post-yielding. This is consistent with the DEM study of simple shear mimicking the condition of faut gouges Mair and Abe (2011), where a quicker decay of grain splitting than grain abrasion as a function of shear strain is observed. Finally, it is worth to mention the concept of shape preferred orientation (SPO) of survivor grains in fault gouge (Cladouhos, 1999). The fact that SPO can be identified and used to infer the previous kinematic history of the fault hints that the evolution of grain size, grain shape, and fabric are intimately coupled for granular materials under severe shear. Analysis of fault gouge materials suggest that the grain shape is highly dependent on the mineral (e.g., quartz, feldspar) for larger grains, but the differences diminish for smaller grains (Heilbronner and Keulen, 2006).

Constitutive modeling. In soil mechanics, grain shape is known to impact the mechanical properties of sand including stiffness, strength, and packing density (Alshibli and Cil, 2018; Cho et al., 2006; Liu and Yang, 2018). The identification of a potential grain shape attractor gives another motivation to incorporate grain shape descriptors in soil constitutive models. However, as of now, models that incorporating grain shape and its dynamic evolution during breakage are still rare. To my knowledge, the only attempt is made recently by Buscarnera and Einav (2021). They showed that such shape attractors can be incorporated in CBM via additional shape-related internal state variables (ISVs). The shape ISVs are introduced in the stored elastic energy of the granular assembly in a way similar to the breakage variable. This dependency is motivated by assuming a linear scaling between the elastic strain energy and the surface area of the particles. Perturbation of the grain shape thus causes energy release or gain of the system, which must be balanced with the energy dissipation and the external work input to the material element. By proposing a coevolution law between grain shape and size, the evolution of grain shape towards an ultimate attractor is predicted throughout the course of loading. The theory has a one-descriptor formulation based on aspect ratio and a two-descriptor version using the elongation ratio and flatness ratio (i.e., the Zingg plane) as shape ISVs. The predicted shape evolution path agrees well with experimental and DEM data from sands with different initial grain morphologies (Fig. 9). It is expected that, as growing experimental data on shape attractors emerge, more theoretical developments will attempt to incorporate grain shape in the continuum description of granular materials.

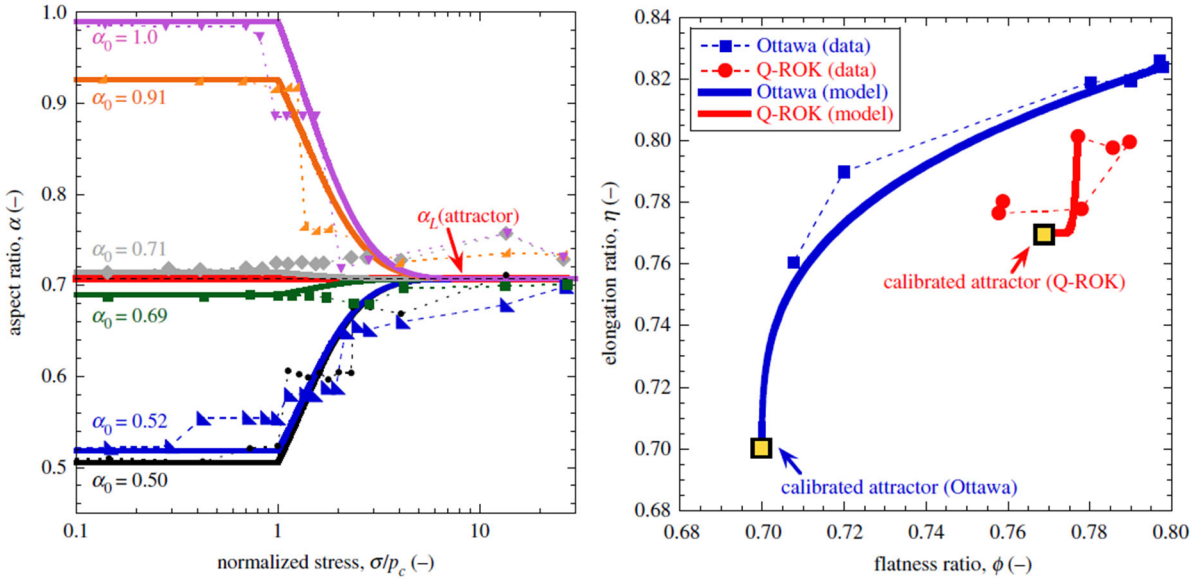

Figure 9: Grain shape evolution predicted by the CBM model of Buscarnera and Einav (2021) compared to DEM data (Left) (Ueda et al., 2013) and experimental data (right) (Seo et al., 2020)

6. Concluding Remarks

Attractors disclose important information for otherwise intangible and complex systems. They provide the backbones for phenomenological macroscopic models to describe the system properties without resorting to the detailed interaction and properties of the constituting subunits. Certainly, much more needs to be done to better quantify these attractors and understand how different initial states evolve towards their steady states for stressed granular materials. It is envisioned that new theories or research methodology for complex systems and self-organization could bring new insights into granular mechanics research, if properly combined with the more traditional continuum mechanics approach.

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References

Alaei, E., Marks, B., Einav, I., 2021. A hydrodynamic-plastic formulation for modelling sand using a minimal set of parameters. Journal of the Mechanics and Physics of Solids 151, 104388.

Alan Bishop, W., Henkel, D.J., 1957. The measurement of soil properties in the triaxial test. Edward Arnold (Publishers) Ltd; London.

Alshibli, K.A., Cil, M.B., 2018. Influence of particle morphology on the friction and dilatancy of sand. Journal of Geotechnical and Geoenvironmental Engineering 144, 04017118.

Anderson, R.S., Bunas, K.L., 1993. Grain size segregation and stratigraphy in aeolian ripples modelled with a cellular automaton. Nature 365, 740-743.

Andò, E., Hall, S.A., Viggiani, G., Desrues, J., Bésuelle, P., 2012. Grain-scale experimental investigation of localised deformation in sand: a discrete particle tracking approach. Acta Geotechnica 7, 1-13.

Anusree, K., Latha, G.M., 2023. Characterization of sand particle morphology: state-of-the-art. Bulletin of Engineering Geology and the Environment 82, 269.

Bak, P., 2013. How nature works: the science of self-organized criticality. Springer Science & Business Media.
Bedau, M.A., 1997. Weak emergence. Philosophical perspectives 11, 375-399.
Ben-Nun, O., Einav, I., 2010. The role of self-organization during confined comminution of granular materials. Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 368, 231-247.
Ben-Nun, O., Einav, I., Tordesillas, A., 2010. Force attractor in confined comminution of granular materials. Physical review letters 104, 108001.
Billi, A., 2005. Grain size distribution and thickness of breccia and gouge zones from thin (< 1 m) strike-slip fault cores in limestone. Journal of Structural Geology 27, 1823-1837.
Buscarnera, G., Einav, I., 2021. The mechanics of brittle granular materials with coevolving grain size and shape. Proceedings of the Royal Society A 477, 20201005.
Casagrande, A., 1936. Characteristics of cohesionless soils affecting the stability of slopes and earth fills. Journal of Boston Society of Civil Engineers 23, 13-32.
Cho, G.-C., Dodds, J., Santamarina, J.C., 2006. Particle shape effects on packing density, stiffness, and strength: natural and crushed sands. Journal of geotechnical and geoenvironmental engineering 132, 591-602.
Cladouhos, T.T., 1999. Shape preferred orientations of survivor grains in fault gouge. Journal of Structural Geology 21, 419-436.
Collins-Craft, N.A., Stefanou, I., Sulem, J., Einav, I., 2020. A Cosserat Breakage Mechanics model for brittle granular media. Journal of the Mechanics and Physics of Solids 141, 103975.
Collins, I., Muhunthan, B., 2003. On the relationship between stress–dilatancy, anisotropy, and plastic dissipation for granular materials. Geotechnique 53, 611-618.
Coop, M., Sorensen, K., Freitas, T.B., Georgoutsos, G., 2004. Particle breakage during shearing of a carbonate sand. Géotechnique 54, 157-164.
Dafalias, Y.F., Taiebat, M., 2016. SANISAND-Z: zero elastic range sand plasticity model. Géotechnique 66, 999-1013.
Das, A., Nguyen, G.D., Einav, I., 2013. The propagation of compaction bands in porous rocks based on breakage mechanics. Journal of Geophysical Research: Solid Earth 118, 2049-2066.
Deng, N., Wautier, A., Thiery, Y., Yin, Z.-Y., Hicher, P.-Y., Nicot, F., 2021. On the attraction power of critical state in granular materials. Journal of the Mechanics and Physics of Solids 149, 104300.
Desrues, J., Chambon, R., Mokni, M., Mazerolle, F., 1996. Void ratio evolution inside shear bands in triaxial sand specimens studied by computed tomography. Géotechnique 46, 529-546.
Domokos, G., Jerolmack, D.J., Kun, F., Török, J., 2020. Plato's cube and the natural geometry of fragmentation. Proceedings of the National Academy of Sciences 117, 18178-18185.
Domokos, G., Kun, F., Sipos, A.A., Szabó, T., 2015. Universality of fragment shapes. Scientific reports 5, 9147.
Einav, I., 2007a. Breakage mechanics-part I: theory. Journal of the Mechanics and Physics of Solids 55, 1274-1297.
Einav, I., 2007b. Breakage mechanics-Part II: Modelling granular materials. Journal of the Mechanics and Physics of Solids 55, 1298-1320.
Einav, I., 2007c. Fracture propagation in brittle granular matter, Proceedings of the royal society of London A: mathematical, physical and engineering sciences. The Royal Society, pp. 3021-3035.
Esna Ashari, S., Das, A., Buscarnera, G., 2018. Model-Based Assessment of the Effect of Surface Area Growth on the Permeability of Granular Rocks. Journal of Engineering Mechanics 144, 04018023.
Foley, B.J., 2018. On the dynamics of coupled grain size evolution and shear heating in lithospheric shear zones. Physics of the Earth and Planetary Interiors 283, 7-25.
Fu, P., Dafalias, Y.F., 2011. Fabric evolution within shear bands of granular materials and its relation to critical state theory. International Journal for numerical and analytical methods in geomechanics 35, 1918-1948.

Gajo, A., Wood, M., 1999. Severn–Trent sand: a kinematic-hardening constitutive model: the q–p formulation. Géotechnique 49, 595-614.
Gao, S., Zhang, Y.D., Sonta, A., Buscarnera, G., 2016. Evolution of the water retention characteristics of granular materials subjected to grain crushing. Journal of Geotechnical and Geoenvironmental Engineering 142.
Hardin, B.O., 1985. Crushing of soil particles. Journal of Geotechnical Engineering 111, 1177-1192.
Heilbronner, R., Keulen, N., 2006. Grain size and grain shape analysis of fault rocks. Tectonophysics 427, 199-216.
Houlsby, G.T., Puzrin, A.M., 2007. Principles of hyperplasticity: an approach to plasticity theory based on thermodynamic principles. Springer Science & Business Media.
Huang, X., O'sullivan, C., Hanley, K., Kwok, C., 2014. Discrete-element method analysis of the state parameter. Géotechnique 64, 954-965.
Imseeh, W.H., Druckrey, A.M., Alshibli, K.A., 2018. 3D experimental quantification of fabric and fabric evolution of sheared granular materials using synchrotron micro-computed tomography. Granular Matter 20, 1-28.
Jefferies, M., 1993. Nor-Sand: a simle critical state model for sand. Géotechnique 43, 91-103.
Jiang, Y., Liu, M., 2015. Applying GSH to a wide range of experiments in granular media. The European Physical Journal E 38, 1-27.
Johnson, C., Kokelaar, B., Iverson, R.M., Logan, M., LaHusen, R., Gray, J., 2012. Grain‐size segregation and levee formation in geophysical mass flows. Journal of Geophysical Research: Earth Surface 117.
Kendall, K., 1978. The impossibility of comminuting small particles by compression. Nature 272, 710-711.
Kenney, T., Lau, D., 1985. Internal stability of granular filters. Canadian geotechnical journal 22, 215-225.
Kruyt, N.P., Rothenburg, L., 2014. On micromechanical characteristics of the critical state of two-dimensional granular materials. Acta mechanica 225, 2301-2318.
Kuwik, B.S., Garcia, M., Hurley, R.C., 2022. Experimental breakage mechanics of confined granular media across strain rates and at high pressures. International Journal of Solids and Structures 259, 112024.
Li, X.S., Dafalias, Y.F., 2012. Anisotropic critical state theory: role of fabric. Journal of Engineering Mechanics 138, 263-275.
Liu, X., Yang, J., 2018. Shear wave velocity in sand: effect of grain shape. Géotechnique 68, 742-748.
Lorenz, E.N., 1963. Deterministic nonperiodic flow. Journal of atmospheric sciences 20, 130-141.
Ma, G., Chen, Y., Yao, F., Zhou, W., Wang, Q., 2019. Evolution of particle size and shape towards a steady state: Insights from FDEM simulations of crushable granular materials. Computers and Geotechnics 112, 147-158.
Mair, K., Abe, S., 2011. Breaking up: comminution mechanisms in sheared simulated fault gouge. Pure and Applied Geophysics 168, 2277-2288.
Manzari, M.T., Dafalias, Y.F., 1997. A critical state two-surface plasticity model for sands. Geotechnique 47, 255-272.
Marachi, N.D., Chan, C.K., Seed, H.B., 1972. Evaluation of properties of rockfill materials. Journal of Soil Mechanics & Foundations Div 98, 95-114.
Marinelli, F., Buscarnera, G., 2019. Anisotropic breakage mechanics: From stored energy to yielding in transversely isotropic granular rocks. Journal of the Mechanics and Physics of Solids 129, 1-18.
Marone, C., Scholz, C., 1989. Particle-size distribution and microstructures within simulated fault gouge. Journal of Structural Geology 11, 799-814.
McDowell, G., 2001. Statistics of soil particle strength. Géotechnique 51, 897-900.
McDowell, G., Bolton, M., 1998. On the micromechanics of crushable aggregates. Géotechnique 48, 667-679.
McDowell, G.R., de Bono, J.P., 2013. On the micro mechanics of one-dimensional normal compression. Géotechnique 63, 895.

McKay, D., Fruland, R., Heiken, G., 1974. Grain size and the evolution of lunar soils, Lunar and Planetary Science Conference Proceedings, pp. 887-906.
Miao, G., Airey, D., 2013. Breakage and ultimate states for a carbonate sand. Géotechnique 63, 1221-1229.
Michikami, T., Nakamura, A.M., Hirata, N., 2010. The shape distribution of boulders on Asteroid 25143 Itokawa: Comparison with fragments from impact experiments. Icarus 207, 277-284.
Mooney, M.A., Finno, R.J., Viggiani, M.G., 1998. A unique critical state for sand? Journal of Geotechnical and Geoenvironmental Engineering 124, 1100-1108.
Nakata, Y., Hyodo, M., Hyde, A.F., Kato, Y., Murata, H., 2001a. Microscopic particle crushing of sand subjected to high pressure one-dimensional compression. Soils and foundations 41, 69-82.
Nakata, Y., Kato, Y., Hyodo, M., HYDE, A.F., Murata, H., 2001b. One-dimensional compression behaviour of uniformly graded sand related to single particle crushing strength. Soils and Foundations 41, 39-51.
Nguyen, G.D., Einav, I., 2010. Nonlocal regularisation of a model based on breakage mechanics for granular materials. International Journal of Solids and Structures 47, 1350-1360.
Nguyen, H., Rahman, M., Fourie, A., 2016. Undrained behaviour of granular material and the role of fabric in isotropic and K 0 consolidations: DEM approach. Géotechnique 67, 153-167.
Nguyen, H., Rahman, M., Fourie, A., 2018. Characteristic Behavior of Drained and Undrained Triaxial Compression Tests: DEM Study. Journal of Geotechnical and Geoenvironmental Engineering 144, 04018060.
Niemunis, A., Herle, I., 1997. Hypoplastic model for cohesionless soils with elastic strain range. Mechanics of Cohesive‐frictional Materials 2, 279-299.
Novák-Szabó, T., Sipos, A.Á., Shaw, S., Bertoni, D., Pozzebon, A., Grottoli, E., Sarti, G., Ciavola, P., Domokos, G., Jerolmack, D.J., 2018. Universal characteristics of particle shape evolution by bed-load chipping. Science advances 4, eaao4946.
Oda, M., 1972a. Deformation mechanism of sand in triaxial compression tests. Soils and Foundations 12, 45-63.
Oda, M., 1972b. The mechanism of fabric changes during compressional deformation of sand. Soils and foundations 12, 1-18.
Papadimitriou, A.G., Chaloulos, Y.K., Dafalias, Y.F., 2019. A fabric-based sand plasticity model with reversal surfaces within anisotropic critical state theory. Acta Geotechnica 14, 253-277.
Petalas, A.L., Dafalias, Y.F., Papadimitriou, A.G., 2020. SANISAND-F: Sand constitutive model with evolving fabric anisotropy. International Journal of Solids and Structures 188, 12-31.
Roscoe, K.H., Schofield, A., Wroth, C., 1958. On the yielding of soils. Geotechnique 8, 22-53.
Sammis, C., King, G., Biegel, R., 1987. The kinematics of gouge deformation. Pure and Applied Geophysics 125, 777-812.
Schofield, A., Wroth, P., 1968. Critical state soil mechanics.
Seo, D., Sohn, C., Cil, M.B., Buscarnera, G., 2020. Evolution of particle morphology and mode of fracture during the oedometric compression of sand. Géotechnique, 1-13.
Shen, X., Buscarnera, G., 2022a. A breakage–damage framework for porous granular rocks in surface-reactive environments. International Journal of Rock Mechanics and Mining Sciences 154, 105111.
Shen, X., Buscarnera, G., 2022b. Fabric-enriched continuum breakage mechanics (F-CBM). Géotechnique, 1-16.
Singh, S., Zurakowski, Z., Dai, S., Zhang, Y., 2021. Effect of grain crushing on hydraulic conductivity of tailings sand. Journal of Geotechnical and Geoenvironmental Engineering 147, 04021143.
Tasiopoulou, P., Gerolymos, N., 2016. Constitutive modelling of sand: a progressive calibration procedure accounting for intrinsic and stress-induced anisotropy. Géotechnique 66, 754-770.
Tengattini, A., Das, A., Einav, I., 2016. A constitutive modelling framework predicting critical state in sand undergoing crushing and dilation. Géotechnique 66, 695-710.

Tengattini, A., Das, A., Nguyen, G.D., Viggiani, G., Hall, S.A., Einav, I., 2014. A thermomechanical constitutive model for cemented granular materials with quantifiable internal variables. Part I—Theory. Journal of the Mechanics and Physics of Solids 70, 281-296.
Theocharis, A.I., Vairaktaris, E., Dafalias, Y.F., Papadimitriou, A.G., 2019. Necessary and sufficient conditions for reaching and maintaining critical state. International Journal for Numerical and Analytical Methods in Geomechanics 43, 2041-2055.
Tsoungui, O., Vallet, D., Charmet, J.-C., 1999. Numerical model of crushing of grains inside two-dimensional granular materials. Powder technology 105, 190-198.
Turcotte, D., 1986. Fractals and fragmentation. Journal of Geophysical Research: Solid Earth 91, 1921-1926.
Ueda, T., Matsushima, T., Yamada, Y., 2013. DEM simulation on the one-dimensional compression behavior of various shaped crushable granular materials. Granular Matter 15, 675-684.
Vaid, Y., Thomas, J., 1995. Liquefaction and postliquefaction behavior of sand. Journal of Geotechnical Engineering 121, 163-173.
Verdugo, R., Ishihara, K., 1996. The steady state of sandy soils. Soils and foundations 36, 81-91.
Vesic, A.S., Clough, G.W., 1968. Behavior of granular materials under high stresses. Journal of Soil Mechanics & Foundations Div.
von Wolffersdorff, P.A., 1996. A hypoplastic relation for granular materials with a predefined limit state surface. Mechanics of Cohesive‐frictional Materials 1, 251-271.
Wang, F., Sassa, K., Wang, G., 2002. Mechanism of a long-runout landslide triggered by the August 1998 heavy rainfall in Fukushima Prefecture, Japan. Engineering Geology 63, 169-185.
Wang, R., Cao, W., Xue, L., Zhang, J.-M., 2021. An anisotropic plasticity model incorporating fabric evolution for monotonic and cyclic behavior of sand. Acta Geotechnica 16, 43-65.
Wang, R., Fu, P., Zhang, J.-M., Dafalias, Y.F., 2017. Evolution of various fabric tensors for granular media toward the critical state. Journal of Engineering Mechanics 143, 04017117.
Wen, Y., Zhang, Y., 2022a. Evidence of a Unique Critical Fabric Surface for Granular Soils. Géotechnique, published online.
Wen, Y., Zhang, Y., 2022b. Relation between void ratio and contact fabric of granular soils. Acta Geotechnica, 1-16.
Wen, Y., Zhang, Y., 2023. Jamming phase diagram in fabric space. Granular Matter, Under Review.
Wiebicke, M., Andò, E., Viggiani, G., Herle, I., 2020. Measuring the evolution of contact fabric in shear bands with X-ray tomography. Acta Geotechnica 15, 79-93.
Wood, D.M., 1990. Soil behaviour and critical state soil mechanics. Cambridge university press.
Wood, D.M., Maeda, K., 2008. Changing grading of soil: effect on critical states. Acta Geotechnica 3, 3-14.
Wroth, C., 1958. Soil behaviour during shear–existence of critical void ratios. Engineering 186, 409-413.
Yang, J., Gu, X., 2013. Shear stiffness of granular material at small strains: does it depend on grain size? Géotechnique 63, 165-179.
Yoshimine, M., Ishihara, K., Vargas, W., 1998. Effects of principal stress direction and intermediate principal stress on undrained shear behavior of sand. Soils and Foundations 38, 179-188.
Zhang, X., Baudet, B., 2013. Particle breakage in gap-graded soil. Géotechnique Letters 3, 72-77.
Zhang, X., Baudet, B.A., Hu, W., Xu, Q., 2017. Characterisation of the ultimate particle size distribution of uniform and gap-graded soils. Soils and foundations 57, 603-618.
Zhang, Y., Buscarnera, G., 2017. A rate-dependent breakage model based on the kinetics of crack growth at the grain scale. Géotechnique 67, 953-967.
Zhang, Y., Buscarnera, G., 2018. Breakage mechanics for granular materials in surface-reactive environments. Journal of the Mechanics and Physics of Solids 112, 89-108.
Zhang, Y., Zhou, X., Wen, Y., 2020. Constitutive Theory for Sand Based on the Concept of Critical Fabric Surface. Journal of Engineering Mechanics 146, 04020019.

Zhang, Y.D., Buscarnera, G., Einav, I., 2016. Grain size dependence of yielding in granular soils interpreted using fracture mechanics, breakage mechanics and Weibull statistics. Geotechnique 66, 149-160.
Zhao, C.-F., Pinzón, G., Wiebicke, M., Andò, E., Kruyt, N.P., Viggiani, G., 2021. Evolution of fabric anisotropy of granular soils: X-ray tomography measurements and theoretical modelling. Computers and Geotechnics 133, 104046.
Zhao, J., Gao, Z., 2015. Unified anisotropic elastoplastic model for sand. Journal of Engineering Mechanics 142, 04015056.
Zhao, J., Guo, N., 2013. Unique critical state characteristics in granular media considering fabric anisotropy. Géotechnique 63, 695-704.
Zhu, F., Zhao, J., 2021. Interplays between particle shape and particle breakage in confined continuous crushing of granular media. Powder Technology 378, 455-467.
Zhu, H., Nguyen, H.N., Nicot, F., Darve, F., 2016. On a common critical state in localized and diffuse failure modes. Journal of the Mechanics and Physics of Solids 95, 112-131.
Zipf, G.K., 2016. Human behavior and the principle of least effort: An introduction to human ecology. Ravenio Books.